Mathematisches Institut
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# PDG I
# (Zentralübung)

## Problem Sheet 8

### Question 1

(a) For $\alpha > 0$, let $g(x) := e^{\alpha |x|^2}$, $x \in \mathbb{R}^N$, and

$$G(t,x) := \frac{1}{(1-4\alpha t)^{N/2}}\, e^{\frac{\alpha}{1-4\alpha t}|x|^2}, \quad x \in \mathbb{R}^N, \quad t \in \left(0, \frac{1}{4\alpha}\right).$$

Prove that $G$ solves

$$\begin{cases} G_t - \Delta G = 0 & \text{in } (0, \frac{1}{4\alpha}) \times \mathbb{R}^N \\ G(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N . \end{cases}$$

(b) Let $\Phi$ be the fundamental solution of the heat equation. Compute $\int_{\mathbb{R}^N} \Phi(t, x-y) g(y)\, \mathrm{d}y$.

### Question 2

Give an alternate (direct) proof that if $\Omega \subset \mathbb{R}^N$ is open and bounded, $T > 0$, and $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation (for $g \in C(\partial' \Omega_T)$)

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial' \Omega_T , \end{cases}$$

then

$$\max_{(t,x) \in \overline{\Omega_T}} u(t,x) = \max_{(t,x) \in \partial' \Omega_T} u(t,x) .$$

*Hint:* Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show that $u_\epsilon$ cannot attain its maximum over $\overline{\Omega_T}$ at a point in the interior $\Omega_T$.

**Question 3**

Let $N = 1$ and $\Phi$ be the fundamental solution of the heat equation. Use properties of the convolution

$$u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y) f(y)\, \mathrm{d}y$$

to prove:

**Weierstrauss'approximation theorem:** A function $f \in C([a,b])$ can be approximated uniformly by polynomials. That is, there exists a sequence of polynomials $p_j$ such that

$$\max_{x\in[a,b]} |f(x) - p_j(x)| \to 0 \quad \text{as } j \to \infty\,.$$

*Hint:* Define $f(x) = f(b)$ for $x > b$ and $f(x) = f(a)$ for $x < a$. Then $u(t,x) \to f(x)$ as $t \to 0$ uniformly for $a \leq x \leq b$. Approximate $\Phi(t, x-y)$ by its truncated power series with respect to $x - y$.

**Deadline for handing in: 0800 Wednesday 10 December**

*Please put solutions in Box 17, 1st floor (near the library)*

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`

# Sheet 8

① $\alpha > 0$, $g(x) := e^{\alpha |x|^2}$, $x \in \mathbb{R}^n$,

$$G(t,x) := \frac{1}{(1-4\alpha t)^{n/2}} e^{\frac{\alpha}{1-4\alpha t}|x|^2}, \quad x \in \mathbb{R}^n, \quad t \in (0, \tfrac{1}{4\alpha})$$

(a) Show
$$\begin{cases} G_t - \Delta G = 0 & \text{in } (0, \frac{1}{4\alpha}) \times \mathbb{R}^n \\ G(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^n \end{cases}$$

Calculation:

$$G_t(t,x) = \frac{\partial}{\partial t}(1-4\alpha t)^{-n/2} \cdot e^{\frac{\alpha}{1-4\alpha t}|x|^2} + \frac{1}{(1-4\alpha t)^{n/2}} \frac{\partial}{\partial t}\left(e^{\frac{\alpha}{1-4\alpha t}|x|^2}\right)$$

$$= (-4\alpha)(-\tfrac{n}{2})(1-4\alpha t)^{-\frac{n}{2}-1} e^{\frac{\alpha}{1-4\alpha t}|x|^2} + (1-4\alpha t)^{-\frac{n}{2}} (4\alpha^2 |x|^2)(1-4\alpha t)^{-2} e^{\frac{\alpha}{1-4\alpha t}|x|^2}$$

$$= \left(2n\alpha(1-4\alpha t)^{-\frac{n}{2}-1} + 4\alpha^2|x|^2(1-4\alpha t)^{-\frac{n}{2}-2}\right) \times e^{\frac{\alpha}{1-4\alpha t}|x|^2}$$

$$\frac{\partial G(t,x)}{\partial x_i} = \cancel{2\alpha} \; 2x_i\alpha(1-4\alpha t)^{-\frac{n}{2}-1} e^{\frac{\alpha}{1-4\alpha t}|x|^2}$$

$$\frac{\partial^2}{\partial x_i^2} G(t,x) = 2\alpha(1-4\alpha t)^{-\frac{n}{2}-1} e^{\frac{\alpha}{1-4\alpha t}|x|^2} + 4x_i^2\alpha^2(1-4\alpha t)^{-\frac{n}{2}-2} e^{\frac{\alpha}{1-4\alpha t}|x|^2}$$

So $\Delta_x G(t,x) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} G =$

$$= \left[2n\alpha(1-4\alpha t)^{-\frac{n}{2}-1} + 4|x|^2\alpha^2(1-4\alpha t)^{-\frac{n}{2}-2}\right] e^{\frac{\alpha}{1-4\alpha t}|x|^2}$$

$$= \frac{\partial}{\partial t} G(t,x)$$

Also, $G(0,x) = \frac{1}{1^{n/2}} e^{\frac{\alpha}{1-0}|x|^2} = e^{\alpha |x|^2} = g(x)$.

(b) $\displaystyle\int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy$ (General formula for soln to heat equation: should be

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} e^{\alpha|y|^2} \, dy$$

<u>Note</u>: $$\frac{-|x-y|^2}{4t} + \alpha|y|^2 = \frac{-|x|^2}{4t} - \frac{|y|^2}{4t} + \frac{2x\cdot y}{4t} + \alpha|y|^2$$

$$= \frac{-1}{4t}\left( |x|^2 + 2y - + 2x\cdot y + (1-4t\alpha)|y|^2 \right)$$

"complete the square in $y$"

$$= -\frac{(1-4t\alpha)}{4t}\left|y - \frac{x}{1-4t\alpha}\right|^2 - \underbrace{\frac{|x|^2}{4t}\left(1 - \frac{1}{1-4t\alpha}\right)}$$

$$= \frac{-|x|^2}{4t}\left(\frac{\cancel{1}-4t\alpha\cancel{-1}}{1-4t\alpha}\right) = \frac{\alpha|x|^2}{1-4t\alpha}$$

Hence $\displaystyle\int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy$

$$= e^{\frac{\alpha|x|^2}{1-4t\alpha}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(1-4t\alpha)}{4t}\left|y - \frac{x}{1-4t\alpha}\right|^2} dy$$

change of variable $z = y - \frac{x}{1-4t\alpha}$

$$= e^{\frac{\alpha}{1-4t\alpha}|x|^2} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|z|^2}{4t}(1-4t\alpha)} dz \qquad w = (1-4t\alpha)^{\frac{1}{2}} z$$

$$= e^{\frac{\alpha}{1-4t\alpha}|x|^2} \frac{1}{(1-4\alpha t)^{\frac{n}{2}}} \underbrace{\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|w|^2} dw}_{=1} = G(t,x)$$

//

② $\Omega \subset \mathbb{R}^n$ open, bounded, $T > 0$.

Recall $\Omega_T := (0,T] \times \Omega$ parabolic cylinder

$\partial' \Omega_T := \bar{\Omega}_T \setminus \Omega_T = \{0\} \times \bar{\Omega} \cup [0,T] \times \partial\Omega$ parabolic boundary

(initial + boundary condits)

<u>Claim</u>: Suppose $u \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial'\Omega_T \end{cases}$$

Then $$\max_{\bar{\Omega}_T} u = \max_{\partial'\Omega_T} u$$

<u>Pf</u>: Let $u_\varepsilon(t,x) := u(t,x) - \varepsilon t$, for $\varepsilon > 0$.

Then note $(u_\varepsilon)_t = u_t - \varepsilon$

$\Delta u_\varepsilon = \Delta u$

So $(u_\varepsilon)_t - \Delta u_\varepsilon = -\varepsilon < 0$ in $\Omega_T$. $(*)$

~~Step~~ $u_\varepsilon \in C(\bar{\Omega}_T)$, & $\bar{\Omega}_T$ is closed, bounded, so

$\exists\, (t_0, x_0) \in \bar{\Omega}_T$ s.t. $u_\varepsilon(t_0,x_0) = \max_{(t,x) \in \bar{\Omega}_T} u(t,x)$

Assume (for a contradiction) that $(t_0, x_0) \notin \partial'\Omega_T$.

So $t_0 \in (0,T]$, $x_0 \in \Omega$.

Consider $h(t) = u_\varepsilon(t, x_0)$, $t \in (0,T]$.

$h(t_0) \geq h(t)$ $\forall\, t \in (0,T]$. $t_0$ maximum of $h$.

Hence $h'(t_0) = 0$, i.e. $(u_\varepsilon)_t(t_0,x_0) = 0$.

Now consider $k(x) := (u_\varepsilon)(t_0, x)$, $x \in \Omega$.

Then $k(x_0) \geq k(x)$ $\forall\, x \in \Omega$.

So $k_{x_i}(x_0) = 0$ $\quad i = 1, \dots, n$

and $k_{x_i x_i}(x_0) \leq 0$. So $\Delta k(x_0) = \Delta(u_\varepsilon)(t_0,x_0) \leq 0$.

Then we have $(t_0, x_0) \in \Omega_T$ and

~~$(u_\varepsilon)_t \overset{(1)}{-} \Delta u_\varepsilon^{*}$~~

$$(u_\varepsilon)_t(t_0, x_0) - \Delta_x u_\varepsilon(t_0, x_0) \leq 0. \quad \text{Contradicts } (*).$$

Hence we must have $(t_0, x_0) \in \partial' \Omega_T$.

ie $$\max_{\bar{\Omega}_T} u_\varepsilon = \max_{\partial' \Omega_T} u_\varepsilon \qquad (2)$$

Now note

$$\max_{\bar{\Omega}_T} u(t,x) = \max_{\bar{\Omega}_T} (u(t,x) - \varepsilon t + \varepsilon t)$$

$$\leq \max_{\bar{\Omega}_T} (u(t,x) - \varepsilon t) + \max_{\bar{\Omega}_T} (\varepsilon t)$$

$$= \max_{\bar{\Omega}_T} (\underbrace{u(t,x) - \varepsilon t}_{u_\varepsilon}) + \varepsilon T$$

$$\overset{(2)}{=} \max_{\partial' \Omega_T} (u(t,x) - \varepsilon t) + \varepsilon T$$

$$\leq \max_{\partial' \Omega_T} u(t,x) + \varepsilon T.$$

$\varepsilon > 0$ arbitrary. So let $\varepsilon \downarrow 0$ to get

$$\max_{\bar{\Omega}_T} u(t,x) \leq \max_{\partial' \Omega_T} u(t,x) \qquad (\text{clearly } \text{"}\geq\text{" holds too as } \partial' \Omega_T \subset \bar{\Omega}_T).$$

$\square$

## ③ Weierstrass Approximation Theorem:

Let $f \in C[a,b]$, $\varepsilon > 0$. Then $\exists$ polynomial $p$ s.t.

$$\max_{x \in [a,b]} |f(x) - p(x)| < \varepsilon.$$

Proof: Extend $f$ continuously to all of $\mathbb{R}$ by $f(x) = \begin{cases} f(b) & x \geq b \\ f(a) & x \leq a. \end{cases}$

Let $u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y) f(y)\, dy$, $x \in \mathbb{R}$, $t > 0$.

Then, for $\exists\, t_0 > 0$ sufficiently small s.t.

$$\max_{[a,b]} |u(t,x) - f(x)| < \varepsilon \quad \forall\, t \in (0, t_0].$$

- Now approximate $u(t_0, x)$ by a polynomial.

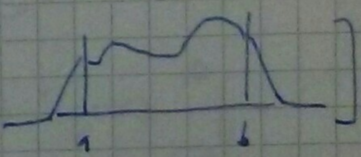
We have $u(t_0, x) = \int_{\mathbb{R}} \Phi(t_0, x-y) f(y)\, dy$ $x \in [a,b]$.

Note $\exists$ an interval $[\alpha, \beta] \supset [a,b]$ s.t.

$\forall\, x \in [a,b]$, $\left| \int_{\mathbb{R} \setminus [\alpha,\beta]} \Phi(t_0, x-y) f(y)\, dy \right| < \varepsilon.$

So $\left| \int_\alpha^\beta \Phi(t_0, x-y) f(y)\, dy - u(t_0, x) \right| < \varepsilon$ $\forall x \in [a,b]$.

[or: extend $f$ out by 0 outside $[a-1, b+1]$]




[More detail: $\Phi(t_0, z) = \frac{1}{(4\pi t_0)^{\frac{1}{2}}} e^{-z^2/4t}$

$\exists\, M$ s.t. $\Phi(t_0, z) < \frac{\varepsilon}{\|f\|_\infty}$ for $|z| > M$.

Then $\left| \int_{\mathbb{R} \setminus [a-M, b+M]} \Phi(t_0, x-y) f(y)\, dy \right| \leq \|f\|_\infty \frac{\varepsilon}{\|f\|_\infty} = \varepsilon.$ ]

· Now approximate $\int_\alpha^\beta \Phi(t_0, x-y) f(y)\, dy$, $x \in [a,b]$

by a polynomial. Let $I$ be interval containing all $x-y$ for $x \in [a,b]$, $y \in [\alpha,\beta]$.

$\Phi(t_0, z) = \sum_{j=0}^\infty a_j z^j$. There exists $J \in \mathbb{N}$ s.t.

converges $\forall z \in \mathbb{R}$

uniformly for bdd $z$.

~~$|\Phi(t_0, z) - \sum_{j=0}^{J} a$~~

$$\sum_{j=J+1}^\infty |a_j z^j| < \frac{\varepsilon}{\|f\|_\infty (\beta-\alpha)} \quad \forall \ z \in I.$$

Then write ~~p(~~ $q(z) = \sum_{j=0}^J a_j z^j$,

Then

$$\left| \int_\alpha^\beta \Phi(t_0, x-y) f(y)\, dy - \int_\alpha^\beta q(x-y) f(y) \right|$$

$$\leq \int_\alpha^\beta |\Phi(t_0, x-y) - q(x-y)|\, |f(y)|\, dy$$

$$\leq \|f\|_\infty \int_\alpha^\beta \sum_{J+1}^\infty |a_j| \, |\underbrace{(x-y)}_{\in I}{}^j| \, dy$$

$$\leq \|f\|_\infty \int_\alpha^\beta \frac{\varepsilon}{(\beta-\alpha)\|f\|_\infty}\, dy = \varepsilon.$$

Now note $\int_\alpha^\beta q(x-y) f(y)\, dy = \sum_{j=0}^J a_j \int_\alpha^\beta (x-y)^j f(y)\, dy$

$$\int_\alpha^\beta (x-y)^j f(y)\, dy = \sum_{k=0}^j \binom{j}{k} x^k \underbrace{\int_a^b (-y)^{j-k} f(y)\, dy}_{\beta^j_k}$$

$$= \sum_{k=0}^j \beta^j_k x^k$$

So $\int_\alpha^\beta q(x-y) f(y)\, dy = \sum_{j=0}^J \sum_{k=0}^j \beta^j_k x^k$ - polynomial in $x$!

$=: p(x)$.

So we have shown: for $x \in [a,b]$:

$$|f(x) - p(x)| \leq |f(x) - u(t_0, x)| + \left| u(t_0, x) - \int_\alpha^\beta \Phi(t_0, x-y) f(y)\, dy \right|$$

$$+ \left| \int_\alpha^\beta \Phi(t_0, x-y) f(y)\, dy - \underbrace{\int_\alpha^\beta q(x-y) f(y)\, dy}_{(= p(x))} \right|$$

$$< 3\varepsilon .$$